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New Upper Bounds for Taxicab and Cabtaxi Numbers

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Abstract

Hardy was surprised by Ramanujan's remark about a London taxi numbered 1729: "it is a very interesting number, it is the smallest number expressible as a sum of two cubes in two different ways". In memory of this story, this number is now called Taxicab(2) $= 1729 = 9^3 + 10^3 = 1^3 + 12^3$, Taxicab($n$) being the smallest number expressible in n ways as a sum of two cubes. We can generalize the problem by also allowing differences of cubes: Cabtaxi(n) is the smallest number expressible in n ways as a sum or difference of two cubes. For example, Cabtaxi(2) $= 91 = 3^3 + 4^3 = 6^3 - 5^3$. Results were only known up to Taxicab(6) and Cabtaxi(9). This paper presents a history of the two problems since Fermat, Frenicle and Viète, and gives new upper bounds for Taxicab(7) to Taxicab(19), and for Cabtaxi(10) to Cabtaxi(30). Decompositions are explicitly given up to Taxicab(12) and Cabtaxi(20).

1 A Fermat problem solved by Frenicle

Our story starts 350 years ago, with letters exchanged between France and England during the reign of Louis XIV and the protectorate of Oliver Cromwell. On August 15th 1657, from Castres (in the south of France), Pierre de Fermat sent to Kenelm Digby some mathematical problems. Translated into English, two of them were:

1. Find two cube numbers of which the sum is equal to two other cube numbers.

2. Find two cube numbers of which the sum is a cube.

These two statements can be written algebraically as follows:

$$x^3 + y^3 = z^3 + w^3 \tag{1}$$

$$x^3 + y^3 = z^3. \tag{2}$$

Fermat asked Digby, who was living in Paris at that time, to pass the problems on to William Brouncker, John Wallis, and Bernard Frenicle de Bessy, defying them to find solutions. Frenicle succeeded in finding several numerical solutions to (1), as announced in October 1657 in a letter sent by Brouncker to Wallis. The first solutions by Frenicle are:

$$1729 = 9^3 + 10^3 = 1^3 + 12^3$$
$$4104 = 9^3 + 15^3 = 2^3 + 16^3$$
$$\dots$$

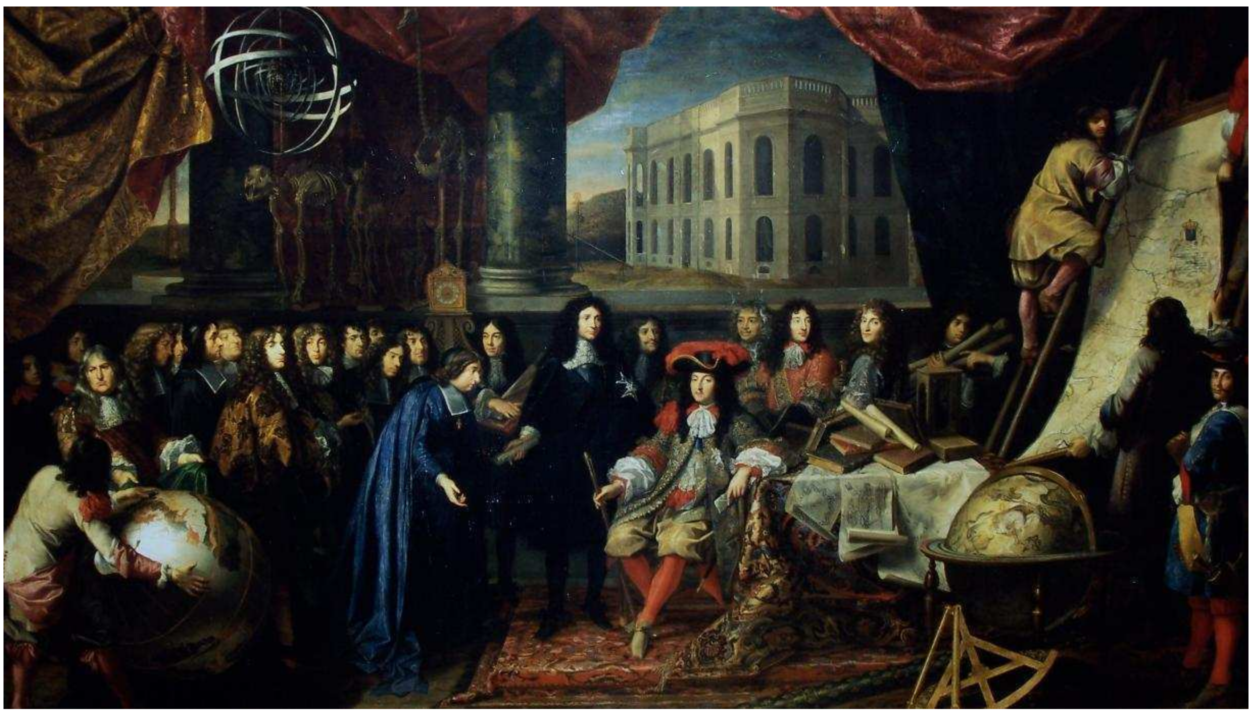
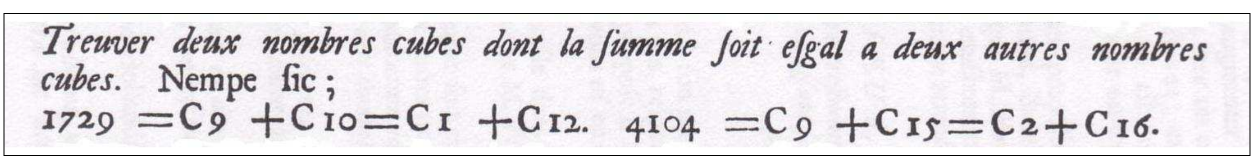


FIGURE 1: Colbert presenting the founding members of the Académie Royale des Sciences to Louis XIV, in 1666. Bernard Frenicle de Bessy (Paris circa 1605 – Paris 1675), one of the founding members, is probably among the people on the left. [Painting by Henri Testelin, Musée du Château de Versailles, MV 2074].



Treuver deux nombres cubes dont la ſumme ſoit eſgal a deux autres nombres cubes. Nempe ſic;
1729 = C9 + C10 = C1 + C12. 4104 = C9 + C15 = C2 + C16.

FIGURE 2: The two smallest of Frenicle's solutions found in 1657,
as published in Wallis's Commercium Epistolicum, Epistola X, Oxford, 1693.

Brouncker added that Frenicle said nothing about equation (2). Slightly later, in February 1658, Frenicle sent numerous other solutions of (1) to Digby without any explanation of the method used. Fermat himself worked on numbers which are sums of two cubes in more than two ways. Intelligently reusing Viète's formulae for solving $x^3 = y^3 + z^3 + w^3$, he proved in his famous comments on Diophantus that it is possible to construct a number expressible as a sum of two cubes in n different ways, for any n, but his method generates huge numbers. We know now that Fermat's method essentially uses the addition law on an elliptic curve. See also Theorem 412 of Hardy & Wright, using Fermat's method [20, pp. 333–334 & 339] .

It was unknown at the time whether equation (2) was soluble; we recognize Fermat's famous last theorem $x^n + y^n = z^n$, when $n = 3$. This particular case was said to be impossible by Fermat in a letter sent to Digby in April 1658, and proved impossible more than one century later by Euler, in 1770. The general case for any n was also said to be impossible by Fermat in his famous note written in the margin of the *Arithmetica* by Diophantus, and proved impossible by Andrew Wiles in 1993–1994. For more details on the Fermat /Frenicle/Digby/Brouncker/Wallis letters, see [1], [12, pp. 551–552], [31, 39, 40, 43].

We will now focus our paper on equation (1). Euler worked on it [16], but the first to have worded it exactly as the problem of the "smallest" solution, which is the true Taxicab problem, seems to have been Edward B. Escott. It was published in 1897 in *L'Intermédiaire des Mathématiciens* [13]:

> Quel est le plus petit nombre entier qui soit, de deux façons différentes, la somme de deux cubes? [In English: What is the smallest integer which is, in two different ways, the sum of two cubes?]

Several authors responded [25] to Escott, stating that Frenicle had found 1729 a long time before. A more complete answer was given by C. Moreau [26], listing all the solutions less than 100,000. C. E. Britton [7] listed all the solutions less than 5,000,000. These two lists are given in the Appendix, figures A1a and A1b.

2 Why is 1729 called a "Taxicab" number?

The problem about the number 1729 is now often called the "Taxicab problem", e.g., [18, p. 212], [22, 37, 44], in view of an anecdote, often mentioned in mathematical books, involving the Indian mathematician Srinivasa Ramanujan (1887–1920) and the British mathematician Godfrey Harold Hardy (1877–1947). Here is the story as related by Hardy and given, for example, in [19, p. xxxv]:

> I remember once [probably in 1919] going to see him [Ramanujan] when he was lying ill in Putney [in the south-west of London]. I had ridden in taxicab No. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways."

As Euler did, Ramanujan worked on parametric solutions of (1). For example, even if a similar formula had been previously found by Werebrusow [45], Ramanujan found [2, p. 107], [29, p. 387] the very nice condition

$$\text{If} \quad m^2 + mn + n^2 = 3a^2b, \quad \text{then} \quad (m + ab^2)^3 + (bn + a)^3 = (bm + a)^3 + (n + ab^2)^3. \quad (3)$$

This equation gives only a small proportion of the solutions. However, with $m = 3, n = 0, a = 1$, and $b = 3$, the equation yields $12^3 + 1^3 = 10^3 + 9^3 = 1729$.

FIGURE 3. Equations handwritten by Ramanujan in two different notebooks: [29, p. 225] (left panel), and [30, p. 341] (right panel).

Euler had published the complete parametric solution in rationals of (1), but as Hardy and Wright [20, p. 200] pointed out, "The problem of finding all integral solutions is more difficult". In 1998, Ajai Choudhry published an interesting paper [11] on the parametric solution in integers of (1).

3 Notation used in this paper

In this paper, Taxicab(n) denotes the smallest integer that can be written in n ways as a sum of two cubes of positive integers. Example:

$$\text{Taxicab}(2) = 1729 = 12^3 + 1^3 = 10^3 + 9^3.$$

Fermat proved that Taxicab(n) exists for any n.

We let $T(n, k)$ denote the kth smallest primitive solution that can be written in n ways as a sum of two cubes of positive integers, so that

$$\text{Taxicab}(n) = T(n, 1) \tag{4}$$

Examples:

$$T(2, 1) = 1729 = \text{Taxicab}(2), \quad T(2, 2) = 4104.$$

When Taxicab(n) is unknown, however, we let $T'(n, k)$ denote the kth smallest known primitive solution (at the time of the article) written in n ways as a sum of two cubes of positive integers, and $T'(n, 1)$ is an upper bound of Taxicab(n):

$$\text{Taxicab}(n) \leq T'(n, 1) \tag{5}$$

We let Cabtaxi(n) denote the smallest integer that can be written in n ways as a sum of two cubes of positive or negative integers. Example:

$$\text{Cabtaxi}(2) = 91 = 3^3 + 4^3 = 6^3 - 5^3.$$

We let $C(n, k)$ denote the kth smallest primitive solution that can be written in n ways as a sum of two cubes of positive or negative integers.

$$\text{Cabtaxi}(n) = C(n, 1) \tag{6}$$

When Cabtaxi(n) is unknown, however, we let $C'(n, k)$ denote the kth smallest known primitive solution written in n ways as a sum of two cubes of positive or negative integers. $C'(n, 1)$ is an upper bound of Cabtaxi(n):

$$\text{Cabtaxi}(n) \leq C'(n, 1). \tag{7}$$

4 1902–2002: from Taxicab(3) to Taxicab(6)

After having asked the question above on Taxicab(2), Escott asked about Taxicab(3) in 1902 [15]. Find the smallest solution of the equation:

$$u^3 + v^3 = w^3 + x^3 = y^3 + z^3. \tag{8}$$

Taxicab(2)	= 1729	10	9	1657	Bernard Frenicle de Bessy (France)
		12	1		
Taxicab(3)	= 87539319	414	255	1957	John Leech (UK)
		423	228		
		436	167		
	(*)	606	-513		
Taxicab(4)	= 6963472309248	16630	13322	1989	E. Rosenstiel, J.A. Dardis, C.R. Rosenstiel (UK)
		18072	10200		
		18948	5436		
		19083	2421		
	(*)	42228	-40884		
Taxicab(5)	= 48988659276962496	331954	231518	1994	John A. Dardis (UK)
		336588	221424		
		342952	205292		
		362753	107839		
		365757	38787		
	(*)	622316	-576920		
	(*)	714700	-681184		
Taxicab(6)	≤ 24153319581254312065344	26224366	18289922	2002	Randall L. Rathbun (USA)
	= 79^3 * Taxicab(5)	26590452	17492496		
	= T'(6, 1)	27093208	16218068		
		28657487	8519281		
		28894803	3064173		
		28906206	**582162**		
		49162964	-45576680		
		56461300	-53813536		
Taxicab(7)	≤ …	…	…	2006	This paper!
…	…	…	…		(and see Fig 6 & 7)
Taxicab(19)	≤ …	…	…		

(*) These supplemental decompositions in differences of cubes were not published by the authors. Of course, they cannot be "counted" as decompositions in this case of Taxicab numbers.

FIGURE 4. History of Taxicab numbers.

The Euler and Werebrusow [46] parametric solutions of (1) and (8) do not help us find the smallest solution. In 1908 André Gérardin [17] suggested that the solution was probably

$$175959000 = 70^3 + 560^3 = 198^3 + 552^3 = 315^3 + 525^3.$$

An important observation for our study and our future "splitting factors" is that Gérardin's solution is equal to $35^3 * T(2,2)$. Two out of its three sums come from the second solution 4104 found by Frenicle as

$$70 = 2 * 35, \quad 560 = 16 * 35,$$
$$315 = 9 * 35, \quad 525 = 15 * 35.$$

But $198^3 + 552^3$ is not a multiple of 35^3 and can be considered as a "new" decomposition. The true Taxicab(3) was discovered more than 50 years after Escott's question, and exactly 300 years after Frenicle's discovery of Taxicab(2). Using an EDSAC machine, John Leech found, and published in 1957 [21], the five smallest 3-way solutions, the smallest of these five being

$$\text{Taxicab}(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3.$$

His results indicated that Gérardin's solution was not Taxicab(3), but $T(3,4)$ = the fourth smallest primitive 3-way solution.

E. Rosenstiel, J. A. Dardis & C. R. Rosenstiel found Taxicab(4) = 6963472309248, and first announced it in 1989 [27]. They gave more detailed results in [36], along with the next three smallest 4-way solutions.

Until now, David W. Wilson was considered to have been the first to have discovered, in 1997, Taxicab(5) = 48988659276962496, see [47], [3, p. 391], [18, p. 212]. But, as kindly communicated to me by Duncan Moore, this number had been previously found three years earlier in 1994 by John A. Dardis, one of the co-discoverers of Taxicab(4), and published in the February 1995 "Numbers count" column of *Personal Computer World* [28]. After Dardis in 1994 and Wilson in 1997, this same number was again found independently by Daniel J. Bernstein [3] in 1998. Bill Butler also confirmed [8] this number in 2006, while computing the fifteen 5-way solutions $< 1.024 * 10^{21}$.

From 1997 to 2002, the best known upper bound of Taxicab(6) was a 6-way solution found by David W. Wilson. In July 2002, Randall L. Rathbun found [32] a better upper bound of Taxicab(6), $2.42*10^{22}$, adding: "I don't believe that this is the lowest value sum for 6 positive cube pairs of equal value". But it seems today that it probably is the lowest value! Calude, Calude & Dinneen claimed in 2003 [9] that this upper bound is the true Taxicab(6) with probability greater than 99%, and further claimed that results in 2005 [10] gave a probability greater than 99.8%, but these claims are not accepted by many mathematicians. And the computations done by Bill Butler proved that Taxicab(6) $> 1.024 * 10^{21}$.

5 Splitting factors

We have remarked that Gérardin's solution was equal to $35^3 * T(2,2)$. It is important to note that $T'(6,1)$ is equal to 79^3*Taxicab(5), as computed by Rathbun. Among the 6 decompositions, only one (underlined in Fig. 4) is a "new" decomposition: the others are 79^3 multiples of the 5 decompositions of Taxicab(5).

Rathbun also remarked that other multiples of Taxicab(5) are able to produce 6-way solutions: $127^3, 139^3$ and 727^3. I add that they are not the only multiples of Taxicab(5) producing 6-way solutions. The next one is 4622^3, which indicates again, as for Gérardin's solution, that non-prime numbers do not have to be skipped as we might initially assume: $79, 127, 139$ and 727 were primes, but $4622 = 2 * 2311$ is not prime.

If N is an n-way sum of two cubes, and if k^3N is an $(n+1)$-way sum of two cubes, then k is called a "splitting factor" of N. This means that this k factor "splits" k^3N into a new $(n+1)$th-way sum of two cubes, the n other sums being directly the k^3 multiples of the already known n ways of N. It was called the "magnification technique" by David W. Wilson.

It is possible that other known 5-way solutions, if they have small splitting factors, may produce smaller 6-way solutions than Rathbun's upper bound. Using the list of 5-way solutions computed by Bill Butler [8], I have computed their splitting factors (Appendix, figure A3). These splitting factors give the smallest known 6-way solutions $< 10^{26}$ (Appendix, figure A4): the first one remains 79^3∗Taxicab(5), which means that it is impossible to do best with this method. We will use this Taxicab(5) number as a basis for our search of upper bounds of Taxicab(n), for larger n.

The method used to find all our decompositions of N into a sum of two cubes is as follows. We first factorize N, then build a list of all its possible pair of factors (r, s) solving $N = rs$, with $r < s$. Because any sum of two cubes can be written as

$$N = rs = x^3 + y^3 = (x + y)(x^2 - xy + y^2), \tag{9}$$

any possible sum of two cubes is an integer solution of the system (10) for one of the possible pairs (r, s):

$$x + y = r, \quad x^2 - xy + y^2 = s. \tag{10}$$

We search for integer solutions of this system by solving the resulting quadratic equation. Of course, most of the pairs (r, s) do not give an integer solution (x, y).

6 Taxicab(7) and beyond

The first idea is to use several of the existing splitting factors together. When we use n factors together, we add n new ways. For example, 127^3∗Taxicab(5) gives $5 + 1 = 6$ way-solutions, and $127^3 * 727^3$∗Taxicab(5) gives $5+2 = 7$ way-solutions. Directly using this idea, the smallest 7-way solution is $79^3 * 127^3$∗Taxicab(5).

The second idea is to check, once a splitting factor is used, if a completely new splitting factor is possible on the new number. In our case, yes it is! A very pleasant surprise: 79^3∗Taxicab(5) has a new splitting factor 101, called a "secondary" factor. And because 101 is smaller than 127, we have found a better 7-way solution $79^3 * 101^3 *$Taxicab(5) smaller than $79^3 * 127^3 *$ Taxicab(5). It is possible that some other $T(5, i)$ could produce a smaller 7-way solution if it has a small secondary factor. This is not the case. For example, using $T(5,6)$, the smallest possible 7-way solution is $25^3 * 367^3 * T(5,6)$, bigger than $79^3 * 101^3 *$Taxicab(5).

Primary splitting factors < 32,000	**Secondary splitting factors < 10,000**	**Ternary**
79	101	None
127	377 = 13*29	2971
		7549
		8063 = 11*733
139	4327	None
727	None	
4622 = 2*2311	None	
14309 = 41*349		
16227 = 3*3*3*601		
23035 = 5*17*271		

Figure 5. Detailed list of splitting factors of Taxicab(5).

Taxicab(7)	≤ 24885189317885898975235988544			
	= 101^3 * T'(6, 1)	=	2.49E+28	**= T'(7, 1)**
Taxicab(8)	≤ 50974398750539071400590819921724352			
	= 127^3 * T'(7, 1)	=	5.10E+34	**= T'(8, 1)**
Taxicab(9)	≤ 136897813798023990395783317207361432493888			
	= 139^3 * T'(8, 1)	=	1.37E+41	**= T'(9, 1)**
Taxicab(10)	≤ 7335345315241855602572782233444632535674275447104			
	= 377^3 * T'(9, 1)	=	7.34E+48	**= T'(10, 1)**
Taxicab(11)	≤ 2818537360434849382734382145310807703728251895897826621632			
	= 727^3 * T'(10, 1)	=	2.82E+57	**= T'(11, 1)**
Taxicab(12)	≤ 73914858746493893996583617733225161086864012865017882136931801625152			
	= 2971^3 * T'(11, 1)	=	7.39E+67	**= T'(12, 1)**

Figure 6. Best upper bounds, for Taxicab$(n), n = 7, 8, \ldots, 12$.

Taxicab(7)	≤ 24885189317885898975235988544	2648660966	1847282122
	= 101^3 * T'(6, 1)	2685635652	1766742096
	= T'(7, 1)	2736414008	1638024868
		2894406187	860447381
		2915734948	459531128
		2918375103	309481473
		2919526806	58798362
		4965459364	-4603244680
		5702591300	-5435167136

Figure 7. Upper bound of Taxicab(7) and its 7 decompositions
(2 more decompositions are differences of cubes)

The best upper bounds using this method were computed in November–December 2006, and are listed in Fig. 6. This search needed some hours on a Pentium IV. They are the current records for the upper bounds of the Taxicab numbers.

Fig. 7 gives the full decomposition of the new upper bound of Taxicab(7). Its new 7th decomposition, which is not 101 times one of the 6 decompositions of $T'(6,1)$ from Fig. 4, is underlined.

The other decompositions of upper bounds up to Taxicab(12) are in the detailed lists of the Appendix. We may continue with the other unused splitting factors of Fig. 5, giving (without explicitly stating their decompositions):

$$\begin{aligned}
\text{Taxicab}(13) &\leq T'(13,1) = 4327^3 * T'(12,1) \simeq 5.99 * 10^{78} \\
\text{Taxicab}(14) &\leq T'(14,1) = 4622^3 * T'(13,1) \simeq 5.91 * 10^{89} \\
\text{Taxicab}(15) &\leq T'(15,1) = 7549^3 * T'(14,1) \simeq 2.54 * 10^{101} \\
\text{Taxicab}(16) &\leq T'(16,1) = 8063^3 * T'(15,1) \simeq 1.33 * 10^{113} \\
\text{Taxicab}(17) &\leq T'(17,1) = 14309^3 * T'(16,1) \simeq 3.91 * 10^{125} \\
\text{Taxicab}(18) &\leq T'(18,1) = 16227^3 * T'(17,1) \simeq 1.67 * 10^{138} \\
\text{Taxicab}(19) &\leq T'(19,1) = 23035^3 * T'(18,1) \simeq 2.04 * 10^{151}.
\end{aligned}$$

7 Cabtaxi numbers

But why should we be limited to sums of positive cubes? We can generalize the problem, allowing sums of positive <u>or negative</u> cubes, these are known as Cabtaxi numbers. Their story starts before that of the Taxicab numbers.

FIGURE 8. François Viète (Fontenay-le-Comte 1540 – Paris 1603)

Sit B 2. D 1. Cubus à radice 6 æquabit ſingulares cubos à radicibus 3, 4, 5. Cum itaque dabuntur cubi ab 6 N, & 3 N: exhibentur cubi abs 4 N & 5 N, & horum ſumma illorum differentiæ, erit æqualis.

FIGURE 9. Formula "$6^3 = 3^3 + 4^3 + 5^3$" by François Viète, as republished in 1646 [41, p. 75].

On 31 July 1589, the French king Henri III was killed by the monk Jacques Clément and was succeeded on the throne by Henri IV. In 1591, François Viète, "one of the most influential men at the court" of Henri IV [42, p. 3] published this very nice formula about his problem XVIII, fourth book of *Zetetica* [41, p. 75] [42, p. 146]:

$$6^3 = 3^3 + 4^3 + 5^3.$$

Moving only one term, we can consider that Viète knew Cabtaxi(2):

$$91 = 3^3 + 4^3 = 6^3 - 5^3.$$

In exactly the same year, 1591, Father Pietro Bongo ("Petrus Bungus" in Latin), canon of Bergamo, independently published this same formula in *Numerorum Mysteria* [12, p. 550]. Bongo is also known to have "demonstrated" that the Antichrist was Martin Luther by using the Hebrew alphabet, the sum of the letters being 666: the number of the Beast. It was an answer to the German mathematician Michael Stifel (1487–1567) who previously proved, using the Latin alphabet, that Pope Leo X was the Antichrist. So strange and mystic the reasoning by some mathematicians at that time ...

Back to mathematics! Viète and Euler worked on parametric solution in rationals of:

$$x^3 = y^3 + z^3 + w^3. \tag{11}$$

In 1756, Euler published [16] the same $x = 6$ solution of Viète and Bongo, and some other solutions. In 1920 H. W. Richmond published [33] a list of $C(2, i)$ numbers, with a solving method.

Euler was probably the first to have mentioned some 3-way solutions, his smallest being

$$87^3 - 79^3 = 20^3 + 54^3 = 38^3 + 48^3.$$

But the first mention of the true Cabtaxi(3) that I have found was by Escott in 1902 [14]:

$$728 = 12^3 - 10^3 = 9^3 - 1^3 = 8^3 + 6^3.$$

Answering Escott's problem in 1904, Werebrusow published [46], [12, p. 562] this 3-way formula:

$$\text{If } m^2 + mn + n^2 = 3a^2bc, \text{ then}$$

$$\begin{aligned} \big((m+n)c + ab^2\big)^3 + \big(-(m+n)b - ac^2\big)^3 &= (-mc + ab^2)^3 + (mb - ac^2)^3 \\ &= (-nc + ab^2)^3 + (nb - ac^2)^3. \end{aligned} \tag{12}$$

Cabtaxi(2)	= 91	4	3	1591	François Viète (France),
		6	-5		Pietro Bongo (Italy) independently
Cabtaxi(3)	= 728	8	6	1902	E. B. Escott (USA)
	= (*) 2^3 * Cabtaxi(2)	9	-1		
		12	-10		
Cabtaxi(4)	= 2741256	108	114	~1992	Randall L. Rathbun (USA)
		140	-14		
		168	-126		
		207	-183		
Cabtaxi(5)	= 6017193	166	113	~1992	Randall L. Rathbun (USA)
		180	57		
		185	-68		
		209	-146		
		246	-207		
Cabtaxi(6)	= 1412774811	963	804	~1992	Randall L. Rathbun (USA)
		1134	-357		
		1155	-504		
		1246	-805		
		2115	-2004		
		4746	-4725		
Cabtaxi(7)	= 11302198488	1926	1608	~1992	Randall L. Rathbun (USA)
	= (*) 2^3 * Cabtaxi(6)	**1939**	**1589**		
		2268	-714		
		2310	-1008		
		2492	-1610		
		4230	-4008		
		9492	-9450		
Cabtaxi(8)	= 137513849003496	44298	36984	1998	Daniel J. Bernstein (USA)
	= (*) 23^3 * Cabtaxi(7)	44597	36547		
		50058	**22944**		
		52164	-16422		
		53130	-23184		
		57316	-37030		
		97290	-92184		
		218316	-217350		
Cabtaxi(9)	= 424910390480793000	645210	538680	2005	Duncan Moore (UK)
	= (*) 5^3 * 67^3 * Cabtaxi(7)	649565	532315		
		752409	**-101409**		
		759780	-239190		
		773850	-337680		
		834820	-539350		
		1417050	-1342680		
		3179820	-3165750		
		5960010	**-5956020**		
Cabtaxi(10)	≤ …	…	…	2006	This paper!
…	…	…	…	-2007	(and see Fig. 12 & 13)
Cabtaxi(30)	≤ …	…	…		

(*) These relationships were unpublished (and unknown?) by the authors

FIGURE 10. History of Cabtaxi numbers.

Werebrusow needed the condition $a^3 = 1$, but his formula is true without this condition. This 3-way formula (12) reuses his previous 2-way formula (3). No example was given by Werebrusow, but we can remark that Cabtaxi(3) can be found, applying $(m, n, a, b, c) =$

$(0,3,1,3,1)$. Another observation is that Cabtaxi(3) can be deduced from Cabtaxi(2), using a splitting factor 2, which adds one new decomposition $9^3 - 1^3$. The two other decompositions of Cabtaxi(2) are 2^3 multiples of Cabtaxi(2).

Cabtaxi(4), Cabtaxi(5), Cabtaxi(6), Cabtaxi(7) were found by Randall L. Rathbun in the beginning of the 1990s [18, p. 211], while Cabtaxi(8) was discovered by Daniel J. Bernstein in 1998 [3].

In the same month, January 2005, there were two nice results on Cabtaxi(9) from two different people: on the 24th, Jaroslaw Wroblewski found an upper bound of Cabtaxi(9) [22], and one week later, on the 31st January 2005, Duncan Moore found the true Cabtaxi(9) [23] Moore's search also proved that Cabtaxi(10) $> 4.6 * 10^{17}$.

Just as Taxicab(5) was a strong basis for Taxicab numbers, we observe in Fig. 10 that Cabtaxi(6) is a strong basis used by bigger Cabtaxi numbers. These interesting relations were never published, and show the strength of splitting factors:

$$\begin{aligned}
\text{Cabtaxi}(7) &= 2^3 * \text{Cabtaxi}(6) \\
\text{Cabtaxi}(8) &= 23^3 * \text{Cabtaxi}(7) \\
\text{Cabtaxi}(9) &= (5 * 67)^3 * \text{Cabtaxi}(7).
\end{aligned}$$

Our method is similar to Taxicab numbers, and uses the splitting factors of Cabtaxi(9) given in Fig. 11a. However, because Jaroslaw Wroblewki's number $C'(9,2) = 8.25 * 10^{17}$ is close to $C(9,1) = \text{Cabtaxi}(9) = 4.25 * 10^{17}$, it is interesting also to analyze its splitting factors, as shown in Fig. 11b.

The best upper bounds up to $C'(20,1)$ using the splitting factors of Cabtaxi(9) were computed in November–December 2006. Three better upper bounds $C'(11,1), C'(17,1), C'(18,1)$ are possible, coming from $C'(9,2)$: they were found later, in February 2007. All these numbers are listed in Fig. 12 and are the current records for the upper bounds of the Cabtaxi numbers.

Fig. 13 gives the full decomposition of the new upper bound of Cabtaxi(10). Its new 10th decomposition, which is not 13 times one of the 9 decompositions of Cabtaxi(9) from Fig 10, is underlined.

Primary splitting factors < 10,000	Secondary splitting factors < 1,000	Ternary splitting factors < 200
13	29	None
	127+	None
23	None	
38 = 2*19	37	None
	436 = 2*2*109	None
43	None	
74 = 2*37	19	None
183 = 3*61	73	None
193	None	
219 = 3*73	61	None
349	None	
661	None	
859	None	
872 = 2*2*2*109	19	None
	37	None
4036 = 2*2*1009	19	None
	37	None
	248 = 2*109	None
4367 = 11*397	439	None
4829 = 11*439	397	None

Figure 11a. Detailed list of splitting factors of Cabtaxi(9) = 424910390480793000.

Primary splitting factors < 10,000	Secondary splitting factors < 1,000	Ternary splitting factors < 300
13	17	None
	74 = 2*37	5
	79	7
	417 = 3*139	None
61	11	None
185 = 5*37	199	None
	291 = 3*97	283
	307	None
	379	None
409	None	
849 = 3*283	485 = 5*97	None
995 = 5*199	37	None
	291 = 3*97	None
	379	None
1021	None	
1153	None	
1455 = 3*5*97	37	None
	199	None
	283	None
	379	None
	481 = 13*37	None
1829 = 31*59	None	
1895 = 5*379	None	
5543 = 23*241	None	
6921 = 3*3*769	None	
8465 = 5*1693	None	

Figure 11b. Detailed list of splitting factors of $C'(9,2) = 825001442051661504$.

Cabtaxi(10)	≤ 933528127886302221000			
	= 13^3 * Cabtaxi(9) = (2*5*13*67)^3 * Cabtaxi(6)	=	9.34E+20	**= C'(10, 1)**
Cabtaxi(11)	≤ 8904950890305189093226944			
	= (13*17)^3 * C'(9, 2) (*)	=	8.90E+24	**= C'(11, 1)**
Cabtaxi(12)	≤ 1912223147184127402358643000			
	= 127^3 * C'(10, 1)	=	1.91E+27	**= C'(12, 1)**
Cabtaxi(13)	≤ 23266019031789278104497609381000			
	= 23^3 * C'(12, 1)	=	2.33E+31	**= C'(13, 1)**
Cabtaxi(14)	≤ 567434938166308703690592195193209000			
	= 29^3 * C'(13, 1)	=	5.67E+35	**= C'(14, 1)**
Cabtaxi(15)	≤ 31136289927061691188910174934641764248000			
	= 38^3 * C'(14, 1)	=	3.11E+40	**= C'(15, 1)**
Cabtaxi(16)	≤ 1577146493675455843791867090964409284453944000			
	= 37^3 * C'(15, 1)	=	1.58E+45	**= C'(16, 1)**
Cabtaxi(17)	≤ 23045156159180392847591977008030799542699242304000			
	= (74*5*79*7*61*11)^3 * C'(11, 1) (**)	=	2.30E+49	**= C'(17, 1)**
Cabtaxi(18)	≤ 181609634582880844694340486417510510845396106201660096000			
	=199^3 * C'(17, 1) (***)	=	1.82E+56	**= C'(18, 1)**
Cabtaxi(19)	≤ 298950477236981197723488725070538575992924211134299879660632000			
	= (43*183*73)^3 * C'(16, 1)	=	2.99E+62	**= C'(19, 1)**
Cabtaxi(20)	≤ 2149172021033860338362430683389430843511963750524516489973424104024000			
	= 193^3 * C'(19, 1)	=	2.15E+69	**= C'(20, 1)**

Three upper bounds derive from C'(9, 2):

(*) because it is smaller than 23^3 * C'(10, 1)

(**) because it is smaller than 43^3 * C'(16, 1)

(***) because it is smaller than (43*183)^3 * C'(16, 1)

FIGURE 12. Best upper bounds for Cabtaxi(10) to Cabtaxi(20).

Cabtaxi(10)	≤ 933528127886302221000	8387730	7002840
	= 13^3 * Cabtaxi(9)	8444345	6920095
	= C'(10, 1)	<u>9773330</u>	<u>-84560</u>
		9781317	-1318317
		9877140	-3109470
		10060050	-4389840
		10852660	-7011550
		18421650	-17454840
		41337660	-41154750
		77480130	-77428260

FIGURE 13. Upper bound of Cabtaxi(10) and its 10 decompositions.

The other decompositions of upper bounds up to Cabtaxi(20) are presented in the detailed lists of the Appendix. We may continue with the other unused splitting factors of Fig. 11a, giving (without explicitly stating their decompositions):

$$\begin{aligned}
\text{Cabtaxi}(21) &\le C'(21,1) = 349^3 * C'(20,1) \simeq 9.14 * 10^{76}\\
\text{Cabtaxi}(22) &\le C'(22,1) = 436^3 * C'(21,1) \simeq 7.57 * 10^{84}\\
\text{Cabtaxi}(23) &\le C'(23,2) = 661^3 * C'(22,1) \simeq 2.19 * 10^{93}\\
\text{Cabtaxi}(24) &\le C'(24,2) = 859^3 * C'(23,1) \simeq 1.39 * 10^{102}\\
\text{Cabtaxi}(25) &\le C'(25,2) = 1009^3 * C'(24,1) \simeq 1.42 * 10^{111}\\
\text{Cabtaxi}(26) &\le C'(26,2) = (4367 * 439)^3 * C'(24,1) \simeq 9.77 * 10^{120}\\
\text{Cabtaxi}(27) &\le C'(27,2) = (4367 * 439)^3 * C'(25,1) \simeq 1.00 * 10^{130}
\end{aligned}$$

and of Fig 11b, giving:

$$\begin{aligned}
\text{Cabtaxi}(21) &\le C'(21,2) = (139 * 283 * 291)^3 * C'(18,1) \simeq 2.72 * 10^{77}\\
\text{Cabtaxi}(22) &\le C'(22,2) = 307^3 * C'(21,1) \simeq 7.88 * 10^{84}\\
\text{Cabtaxi}(23) &\le C'(23,1) = 379^3 * C'(22,1) \simeq 4.29 * 10^{92}\\
\text{Cabtaxi}(24) &\le C'(24,1) = 409^3 * C'(23,1) \simeq 2.94 * 10^{100}\\
\text{Cabtaxi}(25) &\le C'(25,1) = 1021^3 * C'(24,1) \simeq 3.12 * 10^{109}\\
\text{Cabtaxi}(26) &\le C'(26,1) = 1153^3 * C'(25,1) \simeq 4.79 * 10^{118}\\
\text{Cabtaxi}(27) &\le C'(27,1) = 1693^3 * C'(26,1) \simeq 2.32 * 10^{128}\\
\text{Cabtaxi}(28) &\le C'(28,1) = 1829^3 * C'(27,1) \simeq 1.42 * 10^{138}\\
\text{Cabtaxi}(29) &\le C'(29,1) = 2307^3 * C'(28,1) \simeq 1.75 * 10^{148}\\
\text{Cabtaxi}(30) &\le C'(30,1) = 5543^3 * C'(29,1) \simeq 2.97 * 10^{159}.
\end{aligned}$$

8 Unsolved problems

8.1 Are these the true Taxicab and Cabtaxi numbers?

The new upper bounds of Taxicab(7) and Cabtaxi(10) announced in this paper, and detailed in Fig 7 and 13, may have a chance of being the correct Taxicab and Cabtaxi numbers. But the probability decreases as n increases, and is close to 0 for Taxicab(19) and Cabtaxi(30). Who can check if some of these upper bounds are the correct Taxicab and Cabtaxi numbers? Or who will find smaller upper bounds? This is a good subject for mathematical computation.

8.2 Prime versions of Taxicab and Cabtaxi numbers

Our construction with splitting factors generates sums of cubes of non-prime integers: at least $n-1$ decompositions are k^3 multiples. What about sums of two cubes of primes? The 2-way solutions using only sums of cubed primes are rare. For what we can call "the prime version of Taxicab numbers", the smallest 2-way solutions are

$$6058655748 = 61^3 + 1823^3 = 1049^3 + 1699^3 \tag{13a}$$

$$6507811154 = 31^3 + 1867^3 = 397^3 + 1861^3. \tag{13b}$$

For the prime version of Cabtaxi numbers, the smallest 2-way solutions are

$$62540982 = 397^3 - 31^3 = 1867^3 - 1861^3 \tag{14a}$$

$$105161238 = 193^3 + 461^3 = 709^3 - 631^3. \tag{14b}$$

The solution (14a) is just a different arrangement of (13b).

But nobody has succeeded yet (as far as we know) in constructing a 3-way solution using only sums, or sums and differences, of cubed primes. Who will be the first, or who can prove that it is impossible?

An "easier" question: instead of directly searching for a 3-way solution using 6 cubed primes, is there another 3-way solution using at least 4 cubed primes, different from this one

$$68913 = 40^3 + 17^3 = 41^3 - 2^3 = 89^3 - 86^3$$

(the 4 primes used are $17, 41, 2, 89$). See puzzles 90 [34] and 386 [35] of Carlos Rivera.

A supplemental remark: our 3-way problems are unsolved, but are solved for a long time if only coprime pairs are used instead of primes. Several 3-way and 4-way solutions using sums of two coprime cubes are known. The smallest 3-way solution was found by Paul Vojta [18, p. 211] in 1983:

$$15170835645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3.$$

And 3-way, 4-way and 5-way solutions using sums or differences of two coprime cubes are known. It is easy to find the smallest 3-way solution:

$$3367 = 15^3 - 2^3 = 16^3 - 9^3 = 34^3 - 33^3.$$

8.3 Who can construct a 4×4 magic square of cubes?

A 3×3 magic square of cubes, using 9 distinct cubed integers, has been proved impossible [18, p. 270], [4, p. 59]: if z^3 is the number in the centre cell, then any line going through the center should have $x^3 + y^3 = 2z^3$. Euler and Legendre demonstrated that such an equation is impossible with distinct integers.

But the question of 4×4 magic squares of cubes, using 16 distinct positive cubed integers, is still open. A breakthrough was made in 2006 by Lee Morgenstern [5] who found a very nice construction method using Taxicab numbers. If

$$a^3 + b^3 = c^3 + d^3 = u \tag{15}$$

$$\text{and} \quad e^3 + f^3 = g^3 + h^3 = v, \tag{16}$$

then the 4×4 square of cubes in Fig. 14 is semi-magic, its 4 rows and 4 columns having the same magic sum $S = uv$.

$(af)^3$	$(de)^3$	$(ce)^3$	$(bf)^3$
$(bh)^3$	$(cg)^3$	$(dg)^3$	$(ah)^3$
$(bg)^3$	$(ch)^3$	$(dh)^3$	$(ag)^3$
$(ae)^3$	$(df)^3$	$(cf)^3$	$(be)^3$

FIGURE 14. Parametric 4×4 magic square of cubes, Morgenstern's method.

Using $u = \text{Taxicab}(2) = 1729$ and the second smallest 2-way solution $v = T(2,2) = 4104$, both found by Frenicle, which implies $(a,b,c,d,e,f,g,h) = (1,12,9,10,2,16,9,15)$, we find the 4×4 semi-magic square of cubes shown in Fig. 15.

16^3	20^3	18^3	192^3
180^3	81^3	90^3	15^3
108^3	135^3	150^3	9^3
2^3	160^3	144^3	24^3

FIGURE 15. 4×4 semi-magic square of cubes. Magic sum $S = 1729 * 4104 = 7,095,816$.

This is not a full solution of the problem, because this square is only "semi-magic", in that the diagonals each have a wrong sum. The diagonals (and the square) would be fully magic if a third equation is simultaneously true:

$$(ae)^3 + (bf)^3 = (cg)^3 + (dh)^3. \tag{17}$$

Using 2-way lists kindly provided by Jaroslaw Wroblewski, University of Wrocław, I can say that there is no solution to the system of 3 equations (15), (16) and (17), with $a,b,c,d,e,f,g,h < 500,000$ or with $a,b,c,d < 1,000,000$ and $e,f,g,h < 25,000$. But that does not mean that the system is impossible. The first person who finds a numerical solution of this system of 3 equations will directly get a 4×4 magic square of cubes! But perhaps somebody will succeed in constructing a 4×4 magic square of cubes using a different method. Or somebody will prove that the problem is unfortunately impossible.

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10 Appendix

	Moreau's list		Diff of cubes (*)	Comments
T(2, 1)	1729	$= 1^3 + 12^3 = 9^3 + 10^3$		Taxicab(2)
T(2, 2)	4104	$= 2^3 + 16^3 = 9^3 + 15^3$	$= 18^3 - 12^3$	
	13832	$= 2^3 + 24^3 = 18^3 + 20^3$		*non-primitive solution =* 2^3 *T(2, 1)*
T(2, 3)	20683	$= 10^3 + 27^3 = 19^3 + 24^3$		
	32832	$= 4^3 + 32^3 = 18^3 + 30^3$	$= 36^3 - 24^3$	*non-primitive solution =* 2^3 *T(2, 2)*
T(2, 4)	39312	$= 2^3 + 34^3 = 15^3 + 33^3$		
T(2, 5)	40033	$= 9^3 + 34^3 = 16^3 + 33^3$		
	46683	$= 3^3 + 36^3 = 27^3 + 30^3$	$= 46^3 - 37^3$	*non-primitive solution =* 3^3 *T(2, 1)*
T(2, 6)	64232	$= 17^3 + 39^3 = 26^3 + 36^3$		
T(2, 7)	65728	$= 12^3 + 40^3 = 31^3 + 33^3$	$= 76^3 - 72^3$	

	Leech's list			
T(3, 1)	87539319	$= 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$	$= 606^3 - 513^3$	Taxicab(3)
T(3, 2)	119824488	$= 11^3 + 493^3 = 90^3 + 492^3 = 346^3 + 428^3$	$= 648^3 - 534^3$	
T(3, 3)	143604279	$= 111^3 + 522^3 = 359^3 + 460^3 = 408^3 + 423^3$	$= 3996^3 - 3993^3$	
T(3, 4)	175959000	$= 70^3 + 560^3 = 198^3 + 552^3 = 315^3 + 525^3$	$= 630^3 - 420^3$	Gérardin's solution = 35^3 T(2, 2)
T(3, 5)	327763000	$= 300^3 + 670^3 = 339^3 + 661^3 = 510^3 + 580^3$		

(*) These supplemental decompositions in differences of cubes were not published by the authors.

FIGURE A1a. Smallest 2-way solutions[1] listed by Moreau.
FIGURE A1b. Smallest 3-way solutions listed by Leech.

n	Taxicab(n) splitting factors < 10,000
2	None
3	794
4	341, 485, 695, 2551
5	79, 127, 139, 727, 4622

FIGURE A2. Splitting factors of Taxicab numbers.

[1] All these solutions were previously found by Frenicle.

Smallest 5-way solutions		Splitting factors < 10,000
T(5, 1)	(*) 48988659276962496	79, 127, 139, 727, 4622
T(5, 2)	490593422681271000	139, 377, 1139, 1297
T(5, 3)	6355491080314102272	109, 6159
T(5, 4)	27365551142421413376	67, 6159
T(5, 5)	47893568195858112000	127, 349, 1961, 3197, 5983
T(5, 6)	55634997032869710456	25, 367, 907, 2713, 7747
T(5, 7)	68243313527087529096	849, 1829, 5421
T(5, 8)	265781191139199122625	163, 613, 793, 3889
T(5, 9)	276114357544758340608	485, 695, 2551
T(5, 10)	343978135086713831424	579, 949, 1321, 1393, 3739
T(5, 11)	357230299141507244544	65, 349, 1961, 3197, 5983
T(5, 12)	461725779831883749000	803, 851
T(5, 13)	572219233725765415608	59, 1142, 1591, 2435, 8751
T(5, 14)	653115573732974625000	11, 367, 907, 2713, 7747
T(5, 15)	794421645362287488000	139, 341, 2551
T'(5, 16)	(**) 1199962860219870469632	19, 6159
T'(5, 17)	(**) 2337654192461288064000	97, 341, 2551
T'(5, 18)	(**) 7413331235096863544832	65, 127, 1961, 3197, 5983
T'(5, 19)	(**) 9972542662841658461688	8318

(*) Taxicab(5), first found by J. A. Dardis in 1994, later by D. W. Wilson.
(**) These are the 16th-19th known, but may not be the 16th-19th smallest.

FIGURE A3. Splitting factors of the smallest 5-way solutions.

Smallest known 6-way solutions < 10^26		equal to
T'(6, 1)	(*) 24153319581254312065344	= 79^3 T(5, 1)
T'(6, 2)	100347536855722268443968	= 127^3 T(5, 1)
T'(6, 3)	131564874138736741545024	= 139^3 T(5, 1)
T'(6, 4)	869296828638589225875000	= 25^3 T(5, 6) = 11^3 T(5, 14)
T'(6, 5)	1317547017227852341749000	= 139^3 T(5, 2)
T'(6, 6)	(**) 8230545258248091551205888	= 109^3 T(5, 3) = 67^3 T(5, 4) = 19^3 T'(5, 16)
T'(6, 7)	18823431000968427932175168	= 727^3 T(5, 1)
T'(6, 8)	26287287319744419966543000	= 377^3 T(5, 2)
T'(6, 9)	98104370901736427032896000	= 127^3 T(5, 5) = 65^3 T(5, 11)

(*) The upper bound of Taxicab(6) found by Randall L. Rathbun, in 2002.
(**) The solution found by David W. Wilson, in 1997.

FIGURE A4. Smallest 6-way solutions derived from 5-way solutions and splitting factors (other 6-way solutions are possible, if they are not derived from 5-way solutions).

List 1. Upper bounds of Taxicab(7..12) and decompositions.

n	i	Upper bound of Taxicab(n)	a	b
7	1	24885189317885898975235988544	2648660966	1847282122
7	2		2685635652	1766742096
7	3		2736414008	1638024868
7	4		2894406187	860447381
7	5		2915734948	459531128
7	6		2918375103	309481473
7	7		2919526806	58798362
7	*D1*		*4965459364*	*-4603244680*
7	*D2*		*5702591300*	*-5435167136*
8	1	50974398750539071400590819921724352	299512063576	288873662876
8	2		336379942682	234604829494
8	3		341075727804	224376246192
8	4		347524579016	208029158236
8	5		367589585749	109276817387
8	6		370298338396	58360453256
8	7		370633638081	39304147071
8	8		370779904362	7467391974
8	*D1*		*630613339228*	*-584612074360*
8	*D2*		*724229095100*	*-690266226272*
9	1	136897813798023990395783317207361432493888	41632176837064	40153439139764
9	2		46756812032798	32610071299666
9	3		47409526164756	31188298220688
9	4		48305916483224	28916052994804
9	5		51094952419111	15189477616793
9	6		51471469037044	8112103002584
9	7		51518075693259	5463276442869
9	8		51530042142656	4076877805588
9	9		51538406706318	1037967484386
9	*D1*		*87655254152692*	*-81261078336040*
9	*D2*		*100667844218900*	*-95947005451808*
10	1	7335345315241855602572782233444632535674275447104	15695330667573128	15137846555691028
10	2		17627318136364846	12293996879974082
10	3		17873391364113012	11757988429199376
10	4		18211330514175448	10901351979041108
10	5		19262797062004847	5726433061530961
10	6		19404743826965588	3058262831974168
10	7		19422314536358643	2059655218961613
10	8		19426825887781312	1536982932706676
10	9		19429379778270560	904069333568884
10	10		19429979328281886	391313741613522
10	*D1*		*33046030815564884*	*-30635426532687080*
10	*D2*		*37951777270525300*	*-36172021055331616*
11	1	2818537360434849382734382145310807703728251895897826621632	11410505395325664056	11005214445987377356
11	2		12815060285137243042	8937735731741157614
11	3		12993955521710159724	8548057588027946352
11	4		13239637283805550696	7925282888762885516
11	5		13600192974314732786	6716379921779399326
11	6		14004053464077523769	4163116835733008647
11	7		14107248762203982476	2223357078845220136
11	8		14120022667932733461	1497369344185092651
11	9		14123302420417013824	1117386592077753452
11	10		14125159098802697120	657258405504578668
11	11		14125594971660931122	284485090153030494
11	*D1*		*24024464402915670668*	*-22271955089263507160*
11	*D2*		*27590942075671893100*	*-26297059307226084832*
12	1	73914858746493893996583617733225161086864012865017882136931801625152	33900611529512547910376	32696492119028498124676
12	2		38073544107142749077782	26554012859002979271194
12	3		38605041855000884540004	25396279094031028611792
12	4		39334962370186291117816	23546015462514532868036
12	5		40406173326689071107206	19954364747606595397546
12	6		41606042841774323117699	12368620118962768690237
12	7		41912636072508031936196	6605593881249149024056
12	8		41950587346428151112631	4448684321573910266121
12	9		41960331491058948071104	3319755565063005505892
12	10		41965847682542813143520	1952714722754103222628
12	11		41965889731136229476526	1933097542618122241026
12	12		41967142660804626363462	845205202844653597674
12	*D1*		*71376683741062457554628*	*-66169978570201879772360*
12	*D2*		*81972688906821194400100*	*-78128563201768698035872*

List 2. Upper bounds of Taxicab(10..20) and decompositions.

n	i	Upper bound of Cabtaxi(n)	a	b
10	1	933528127886302221000	8387730	7002840
10	2		8444345	6920095
10	3		9773330	-84560
10	4		9781317	-1318317
10	5		9877140	-3109470
10	6		10060050	-4389840
10	7		10852660	-7011550
10	8		18421650	-17454840
10	9		41337660	-41154750
10	10		77480130	-77428260
11	1	8904950890305189093226944	187282914	132686190
11	2		200769660	93302664
11	3		205664368	59039708
11	4		207007164	32487000
11	5		207780664	-40314820
11	6		213359622	-93127734
11	7		214963164	-100935120
11	8		232614213	-154412037
11	9		237739866	-165488778
11	10		250837664	-190171940
11	11		692958539	-686721035
12	1	1912223147184127402358643000	1065241710	889360680
12	2		1072431815	878852065
12	3		1241212910	-10739120
12	4		1242227259	-167426259
12	5		1244819331	-255698331
12	6		1254396780	-394902690
12	7		1277626350	-557509680
12	8		1378287820	-890466850
12	9		1537377310	-1198473220
12	10		2339549550	-2216764680
12	11		5249882820	-5226653250
12	12		9839976510	-9833389020
13	1	2326601903178927810449760938100 0	24500559330	20455295640
13	2		24665931745	20213597495
13	3		27686328930	12689982240
13	4		28547896930	-246999760
13	5		28571226957	-3850803957
13	6		28630844613	-5881061613
13	7		28851125940	-9082761870
13	8		29385406050	-12822722640
13	9		31700619860	-20480737550
13	10		35359678130	-27564884060
13	11		53809639650	-50985587640
13	12		120747304860	-120213024750
13	13		226319459730	-226167947460
14	1	5674349381663087036905921951932 09000	710516220570	593203573560
14	2		715312020605	586194327355
14	3		802903538970	368009484960
14	4		825175080660	177175504170
14	5		827889010970	-7162993040
14	6		828565581753	-111673314753
14	7		830294493777	-170550786777
14	8		836682652260	-263400094230
14	9		852176775450	-371858956560
14	10		919317975940	-593941388950
14	11		1025430665770	-799381637740
14	12		1560479549850	-1478582041560
14	13		3501671840940	-3486177717750
14	14		6563264332170	-6558870476340
15	1	311362899270616911889101749346417642480 00	26999616381660	22541735795280
15	2		27181856782990	22275384439490
15	3		30510334480860	13984360428480
15	4		31356653065080	6732669158460
15	5		31459782416860	-272193735520
15	6		31485492106614	-4243585960614
15	7		31551190763526	-6480929897526
15	8		31793940785880	-10009203580740
15	9		32382717467100	-14130640349280
15	10		33289123673715	-17918953469235
15	11		34934083085720	-22569772780100
15	12		38966365299260	-30376502234120
15	13		59298222894300	-56186117579280
15	14		133063529955720	-132474753274500
15	15		249404044622460	-249237078100920
16	1	1577146493675455843791867090964409284453944000	998985806121420	834044224425360
16	2		1005728700970630	824189224261130
16	3		1128882375791820	517421335853760
16	4		1160196163407960	249108758863020
16	5		1164011949423820	-10071168214240
16	6		1164963207944718	-157012680542718
16	7		1167394058250462	-239794406208462
16	8		1176375809077560	-370340532487380
16	9		1198160546282700	-522833692923360
16	10		1231697575927455	-663001278361695
16	11		1292561074171640	-835081592863700
16	12		1441755516072620	-1123930582662440
16	13		1610274784302639	-1374764111814639
16	14		2194034247089100	-2078886350433360
16	15		4923350608361640	-4901565871156500
16	16		9227949651031020	-9221771889734040

LIST 2 (cont'd). Upper bounds of Taxicab(10..20) and decompositions.

17	1	230451561591803928475919770080307995426992423040000	25712691169505340	18216926216388900
17	2		27564331168974600	12809831572377840
17	3		28236341831778080	8105756932753480
17	4		28420698739272840	4460247761970000
17	5		28457028345165420	836124103875060
17	6		28526895114557840	-5534955079854200
17	7		28870628936847005	-10062589409548445
17	8		29292848724728820	-12785814853659540
17	9		29513004313632840	-13857716720047200
17	10		31936375255815030	-21199739663572470
17	11		32640093122096460	-22720502099475180
17	12		34438333163227840	-26109335111721400
17	13		35389267534737480	-27709744552045920
17	14		57295758308286960	-54853115936914680
17	15		81636131772363168	-80466823575306168
17	16		95138571512074090	-94282203941775850
17	17		127480343199333960	-127005894471487680
18	1	1816096345828808446943404864175105108453961062016600960000	5116825542731562660	3625168317061391100
18	2		5485301902625945400	2549156482903190160
18	3		5619032024523837920	1613045629617942520
18	4		5655719049115295160	887589304632030000
18	5		5662948640687918580	166388696671136940
18	6		5676852127797010160	-1101456060890985800
18	7		5697455371523153238	-1494117880642625238
18	8		5745255158432553995	-2002455292500140555
18	9		5829276896221035180	-2544377155878248460
18	10		5873087858412935160	-2757685627289392800
18	11		6355338675907190970	-4218748193050921530
18	12		6495378531297195540	-4521379917795560820
18	13		6853228299482340160	-5195757687232558600
18	14		7042464239412758520	-5514239165857138080
18	15		11401855903349105040	-10915770071446021320
18	16		16245590222700270432	-16012897891485927432
18	17		18932575730902743910	-18762158584413394150
18	18		25368588296667458040	-25274172999826048320
19	1	298950477236981197723488725070538575992924211134299879660632000	573854409510970140540	479105862146230522320
19	2		577727777799465785310	473444785416890733810
19	3		648471805302725705340	297225959903826333120
19	4		666004543444250247510	152350176313334063610
19	5		666459603519578318520	143097288114996619740
19	6		668414091503088701680	68268319603456235900
19	7		668651532191170889340	-5785251655483382880
19	8		668763496903121942140	-53179564334639186080
19	9		669197970282139973766	-90193893172917299766
19	10		670594340639220639894	-137746779319170285894
19	11		675753790639086333720	-212737304460453105060
19	12		688267749724995339900	-300335018061816148320
19	13		707532660423039467835	-380852465338256990715
19	14		742494905763934366680	-479701764959845236900
19	15		828197713386207614940	-645627312112864046280
19	16		925001416270455039243	-789715372098465783243
19	17		1260334450795121336700	-1194189238483888018320
19	18		2828154753415435396680	-2815640794329526390500
19	19		5300875713689306035740	-5297326979023152735480
20	1	2149172021033860338362430683389430843511963750524516489973424104024000	110753901035617237124220	92467431394222490807760
20	2		111501461115296896564830	91374843585459911625330
20	3		125155058423426061130620	57364610261438482292160
20	4		128538876884740297769430	29403584028473474276730
20	5		128626703479278615474360	27617776606194347609820
20	6		129003919660096119424240	13175785683467053528700
20	7		129049745712895981642620	-1116553569508292895840
20	8		129071354902302534833020	-10263655916585362913440
20	9		129155208264453014936838	-17407421382373038854838
20	10		129424707743369583499542	-26585128408599865177542
20	11		130420481593343662407960	-41058299760867449276580
20	12		132835675696924100600700	-57964658485930516625760
20	13		136553803461646617292155	-73504525810283599207995
20	14		143301516812439332769240	-92582440637250130721700
20	15		159842158683538069683420	-124606071237782760932040
20	16		178073220660515627641194	-151793906580106714663194
20	17		178525273340197822573899	-152415066815003896165899
20	18		243244549003458417983100	-230478523027390387535760
20	19		545833867409179031559240	-543418673305598593366500
20	20		1023069012742036064897820	-1022384106951468477947640

List 3. Upper bounds of Taxicab(13..19).

13	5988146776742829080553965820313279739849705084894534523771076163371248442670016
14	591265120715306076227178149379165201865336472346517495072293984450950965960475614042157568
15	254361007450418467683691805874871224061387429621450309607817376336263717732850126485501423293412997632
16	1333338795750670442554964073425419093729210607136222568896962244262422572993460082081694341861362060034829246807 04
17	390632494425592857308683115941747067399542361030496398592266116103976300384768055853952870509972688279894938439376270144482816
18	1669102760262770599073633207770749663711833995754118800009115774272350540676865796358235477762975575548685960795354954969975483624898697728
19	20400824749409517528805616329601248054238975120047899653306832282155305931553802798551427446267572330642814901020208524571911529017539708401834884288000

List 4. Upper bounds of Taxicab(21..30).

21	91358184165546882852435964463963107093539643175395184915343307223685301176000
22	757193586443033690582446685592330743143933123595448645924685617987724492138586265 6000
23	4290958509201630386098377358638838836434240004572859495441380491711971566655172462496952 96000
24	293578494624502994440321369138559612153600445801825576286084194143934759097432357021871690333619 84000
25	31246506296467361807393767238185877428672464177086497353619597318957704158666049984364792262467344862165824000
26	4789491285250967697893538843903291484293985881436343001451592076578457363463189029474506947673318138714465782347 2448000
27	232412917094127964459115806855336482758674965865869951188239880040662691393903205426740733767950588479203326262786046468585536000
28	14220058339037733607720187711173708351128859481217507416693218403566577929284336075168437905172873812406535056473363934236754935255079040 00
29	174599968771160245579288557178174329532287219412331267334689465851562923091100834699719133990753950650302246316479163002022953293056585944949134720 00
30	29735743197506017957555813470209270749990607839871779215457304201897336613566600659342250760095651263146947271255145747139472942729608236996870255748579303040 00

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Keywords: Taxicab number, Cabtaxi number, Hardy–Ramanujan number, Bernard Frenicle de Bessy, François Viète, sum of two cubes, difference of two cubes, magic square of cubes.

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